

THE TUNNEL NUMBER OF THE SUM OF n KNOTS IS AT LEAST n

MARTIN SCHARLEMANN

JENNIFER SCHULTENS

ABSTRACT. We prove that the tunnel number of the sum of n knots is at least n .

1. INTRODUCTION

In [5], Norwood showed that tunnel number 1 knots are prime. This led to the more general conjecture, see for instance [4, Problem 1.70B], that the tunnel number of a sum of n knots is at least n . Here we prove this conjecture. The idea is to show that the splitting surface of a Heegaard splitting corresponding to a tunnel system realizing the tunnel number of the sum of n knots intersects each individual knot complement essentially. Then a sophisticated Euler characteristic argument, based on the idea of untelescoping the Heegaard splitting, yields the result.

We wish to thank MSRI, where part of this work was carried out.

2. PRELIMINARIES

For standard definitions concerning knots, see [1] or [6] and for those concerning 3-manifolds, see [2] or [3].

Definition 1. *Let N be a submanifold of M , we denote an open regular neighborhood of N in M by $\eta(N)$.*

Date: February 1, 2008.

1991 *Mathematics Subject Classification.* 57M25, 57M50, 57N10.

The first author is supported by an NSF grant.

The second author is supported by an NSF postdoctoral fellowship.

Definition 2. Let K be a knot in S^3 . Denote the complement of K , $S^3 - \eta(K)$, by $C(K)$.

Remark 1. Let $K = K_1 \# K_2$ be the sum of two knots. Then the decomposing sphere gives rise to a decomposing annulus A properly embedded in $C(K)$ such that $C(K) = C(K_1) \cup_A C(K_2)$. If $K = K_1 \# \dots \# K_n$, then we may assume that the decomposing spheres are nested, so that $C(K) = C(K_1) \cup_{A_1} \dots \cup_{A_{n-1}} C(K_n)$.

Definition 3. A tunnel system for a knot K is a collection of disjoint arcs $\mathcal{T} = t_1 \cup \dots \cup t_n$ properly embedded in $C(K)$ such that $C(K) - \eta(\mathcal{T})$ is a handlebody. The tunnel number of K , denoted by $t(K)$, is the least number of arcs required in a tunnel system for K .

Definition 4. A compression body is a 3-manifold W obtained from a connected closed orientable surface S by attaching 2-handles to $S \times \{0\} \subset S \times I$ and capping off any resulting 2-sphere boundary components. We denote $S \times \{1\}$ by $\partial_+ W$ and $\partial W - \partial_+ W$ by $\partial_- W$.

Definition 5. A set of defining disks for a compression body W is a set of disks $\{D_1, \dots, D_n\}$ properly embedded in W with $\partial D_i \subset \partial_+ W$ for $i = 1, \dots, n$ such that the result of cutting W along $D_1 \cup \dots \cup D_n$ is homeomorphic to $\partial_- W \times I$.

Definition 6. A Heegaard splitting of a 3-manifold M is a decomposition $M = V \cup_S W$ in which V, W are compression bodies such that $V \cap W = \partial_+ V = \partial_+ W = S$ and $M = V \cup W$. We call S the splitting surface or Heegaard surface.

Definition 7. Let $M = V \cup_S W$ be an irreducible Heegaard splitting. We may think of M as being obtained from $\partial_- V \times I$ by attaching all 1-handles dual to 2-handles in V followed by all 2-handles in W , followed, perhaps, by 3-handles. An untelescoping of $M = V \cup_S W$ is a rearrangement of the order in which the 1-handles (of V) and the 2-handles (dual to the 1-handles of W) are attached. This rearrangement is chosen so that M is decomposed into submanifolds M_1, \dots, M_m , such that $M_i \cap M_{i+1} = F_i$ and F_i is an incompressible surface in M , and such that the M_i inherit, from a subcollection of the original 1-handles and 2-handles, strongly irreducible Heegaard splittings $M_1 = V_1 \cup_{S_1} W_1, \dots, M_m = V_m \cup_{S_m} W_m$. Unless M is a lens space or $S^1 \times S^2$, no S_1, \dots, S_m is a torus. For details see

[8] and [7]. We denote the untelescoping of $M = V \cup_S W$ by $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \dots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$. For convenience, we will occasionally denote $\partial_- V = \partial_- V_1$ by F_0 .

Lemma 2. $\chi(S) = \sum_{i=1}^m \chi(S_i) - \sum_{i=1}^{m-1} \chi(F_i)$.

Proof. Let $M = V \cup_S W$ be a Heegaard splitting, then

$$\chi(S) = \chi(\partial_- V) - 2(\#(1\text{-handles attached in } V) - \#(0\text{-handles attached in } V))$$

and in an untelescoping,

$$\chi(S_i) =$$

$$\chi(\partial_- V_i) - 2(\#(1\text{-handles attached in } V_i) - \#(0\text{-handles attached in } V_i)) =$$

$$\chi(F_{i-1}) - 2(\#(1\text{-handles attached in } V_i) - \#(0\text{-handles attached in } V_i)).$$

So, since 1-handles are merely reordered in an untelescoping,

$$\chi(S) =$$

$$\chi(\partial_- V) - 2 \sum_{i=1}^m \#((1\text{-handles attached in } V_i) - \#(0\text{-handles attached in } V_i)) =$$

$$\chi(\partial_- V) - \sum_{i=1}^m \chi(F_{i-1}) + \sum_{i=1}^m \chi(S_i).$$

□

Lemma 3. Let P be a properly embedded incompressible surface in an irreducible 3-manifold M and let $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \dots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$ be an untelescoping of a Heegaard splitting $M = V \cup_S W$. Then $(\cup_{i=1}^{m-1} F_i) \cup (\cup_{i=1}^m S_i)$ can be isotoped to intersect P only in curves essential in P .

Remark 4. This lemma demonstrates the advantage of working with untelescoped Heegaard splittings rather than Heegaard splittings. It is a deep fact that the splitting surface of a strongly irreducible Heegaard splitting can be isotoped to intersect a properly embedded incompressible surface only in curves essential in this surface. This fact is proven for instance in [9, Lemma 6].

Proof. Here $(\cup_{i=1}^{m-1} F_i)$ may be isotoped to intersect P only in curves essential in P by a standard innermost disk argument, since both are incompressible. Then $P_i = P \cap M_i$ is a properly embedded incompressible surface in M_i . It follows that each S_i may be isotoped in M_i to intersect P_i only in curves essential in P_i , by [9, Lemma 6]. Note that the latter isotopies fix $(\cup_{i=1}^{m-1} F_i)$. \square

Lemma 5. *Let K be a prime knot and let A be an annulus properly embedded in $C(K)$ such that the components of ∂A are meridians. Then A is boundary parallel.*

Proof. In S^3 , A can be extended to a sphere by adding two meridian disks. This sphere intersects K in two points. Since K is prime, one side of the sphere contains a single unknotted arc. \square

Lemma 6. *Let P be an incompressible surface in a compression body W . Then the result of cutting W along P is a collection of compression bodies.*

Proof. This is [9, Lemma 2]. \square

Remark 7. *In the above lemma, P need not be connected.*

Lemma 8. *If \mathcal{A} is a collection of incompressible annuli in a compression body W , then in any component X of $W - \mathcal{A}$, $\chi(\partial_+ W \cap X) \leq \chi(\partial_- W \cap X)$.*

Proof. Let \mathcal{D} be a set of defining disks for W . We argue by induction on the pair $(\|\chi(\partial_- W) - \chi(\partial_+ W)\|, |\mathcal{A} \cap \mathcal{D}|)$. If $\|\chi(\partial_- W) - \chi(\partial_+ W)\| = 0$, then $(\mathcal{D} = \emptyset)$ and all annuli are spanning annuli and the result follows.

To complete the inductive step, suppose there is a disk D in \mathcal{D} such that $D \cap \mathcal{A} = \emptyset$. The result of cutting W along D is a compression body W' with $\|\chi(\partial_- W') - \chi(\partial_+ W')\| < \|\chi(\partial_- W) - \chi(\partial_+ W)\|$, or two compression bodies W' and W'' with $\|\chi(\partial_- W') - \chi(\partial_+ W')\| < \|\chi(\partial_- W) - \chi(\partial_+ W)\|$ and $\|\chi(\partial_- W) - \chi(\partial_+ W'')\| < \|\chi(\partial_+ W)\|$. The components of $W - \mathcal{A}$ can be obtained from the components of $W' - \mathcal{A}$ or of $W' - \mathcal{A}$ and $W'' - \mathcal{A}$ by attaching a 1-handle either to a single component or so as to connect two components. In both cases, the result follows from the inductive hypotheses.

If there is no such disk, consider $\mathcal{D} \cap \mathcal{A}$. If there is an arc α in $\mathcal{D} \cap \mathcal{A}$ that is inessential in \mathcal{A} , then we may assume that α is outermost in \mathcal{A} , and we may cut the disk D in \mathcal{D} containing α along α and paste on two copies of the disk cut off

by α in \mathcal{A} to obtain a new disk D' . Replacing D by D' in \mathcal{D} produces a new set of defining disks \mathcal{D}' with $|\mathcal{A} \cap \mathcal{D}'| < |\mathcal{A} \cap \mathcal{D}|$.

If all arcs in $\mathcal{D} \cap \mathcal{A}$ are essential in \mathcal{A} , let β be an arc in $\mathcal{D} \cap \mathcal{A}$ that is outermost in \mathcal{D} . Let A be the annulus in \mathcal{A} that gives rise to β . Cutting and pasting A along β and the outermost disk cut off in \mathcal{D} yields a disk D' disjoint from \mathcal{A} . If D' is inessential, then A is inessential and can be ignored. (Since cutting along A does not alter any components or their Euler characteristics.) If D' is essential, the result follows as above. This completes the inductive step. \square

3. THE COMBINATORICS

In the following, we consider a tunnel system \mathcal{T} , realizing the tunnel number of $K_1 \# \dots \# K_n$. We also consider the Heegaard splitting $C(K_1 \# \dots \# K_n) = V \cup_S W$ corresponding to \mathcal{T} and an untelescoping $C(K_1 \# \dots \# K_n) = (V_1 \cup_{S_1} W_1) \cup_{F_1} \dots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$ of $C(K_1 \# \dots \# K_n) = V \cup_S W$. Set $M_i = V_i \cup W_i$. By Remark 1, $C(K_1 \# \dots \# K_n) = C(K_1) \cup_{A_1} \dots \cup_{A_{n-1}} C(K_n)$. We will always assume that $\partial_- V_1 = \partial C(K_1 \# \dots \# K_n)$ and that $\cup_{i=1}^{m-1} F_i$ and $\cup_{i=1}^m S_i$ intersect $\cup_{j=1}^{n-1} A_j$ only in curves essential in $\cup_{j=1}^{n-1} A_j$. We will, furthermore, assume that, subject to these constraints, the number of intersections of $\cup_{i=1}^{m-1} F_i$ and $\cup_{i=1}^m S_i$ with $\cup_{j=1}^{n-1} A_j$ is minimal.

Definition 8. Set $S_{ij} = S_i \cap C(K_j)$, $F_{ij} = F_i \cap C(K_j)$ and $A_{ij} = M_i \cap A_j$.

Lemma 9. For all i, j , $\chi(S_{ij})$ and $\chi(F_{ij})$ are even.

Proof. Here F_i is separating, so $F_i \cap A_{j-1}$ is separating. Since $\partial A_{j-1} \subset \partial C(K_1 \# \dots \# K_n)$ which is a torus, hence connected, both components lie on one side of F_i , hence $|F_i \cap A_{j-1}|$ is even. The same is true for $|F_i \cap A_j|$. Thus $\chi(F_i \cap C(K_j)) = 2 - 2(\text{genus}(F_i \cap C(K_j))) - |F_i \cap (A_{j-1} \cup A_j)|$ is even. Similarly for S_i . \square

Definition 9. Set $x_{ij} = -1/2\chi(F_{ij})$ and $y_{ij} = -1/2\chi(S_{ij})$.

| | | | | | |
|-------|---|-----|----------|-----|---|
| | 1 | ... | j | ... | n |
| 1 | | | | | |
| ... | | ... | | ... | |
| i | | | x_{ij} | | |
| ... | | ... | | ... | |
| $m-1$ | | | | | |

fig. 1a

| | | | | | |
|-----|---|-----|----------|-----|---|
| | 1 | ... | j | ... | n |
| 1 | | | | | |
| ... | | ... | | ... | |
| i | | | y_{ij} | | |
| ... | | ... | | ... | |
| m | | | | | |

fig. 1b

Lemma 10. *Under the assumptions above, $y_{ij} \geq \max\{x_{i-1j}, x_{ij}\}$.*

Proof. This follows from Lemma 8. □

Lemma 11. *For all j , there is an i , such that $y_{ij} > 0$.*

Proof. Suppose $y_{ij} = 0$ for $i = 1, \dots, m$. Then $x_{ij} = 0$ for $i = 1, \dots, m-1$. So

$$G_j = (\cup_{i=1}^{m-1} F_{ij}) \cup (\cup_{i=1}^m S_{ij}) \subset C(K_j)$$

is a collection of annuli and tori. Since the tori arise only in $\cup_{i=1}^{m-1} F_{ij}$, they are incompressible separating tori. Thus if a torus component T of F_i is in $C(K_j)$, then so is a component of $S_{i'}$, which cannot be a torus, for some i' . But this would contradict $y_{i'j} = 0$. Hence G_j consists entirely of annuli. By Lemma 5, the annuli are all boundary parallel. Hence cutting $C(K_j)$ along the annular components of G_j yields a copy of $C(K_j)$. By Lemma 6, all components of $C(K_j)$ cut along G are compression bodies, a contradiction. □

Lemma 12. *For all j ,*

$$\sum_{i=1}^m y_{ij} > \sum_{i=1}^{m-1} x_{ij}.$$

Proof. This follows by comparing the tables in fig. 1. By Lemma 10, the largest value encountered in a given column of the table in fig. 1a occurs one time more often in the corresponding column of the table in fig. 1b. If the largest value encountered in a column in the table in fig. 1a is zero, then by Lemma 11, there must be nonzero entries in the corresponding column of the table in fig. 1b. \square

Remark 13. *Since all numbers involved are integers, it follows that $\sum_{i=1}^m y_{ij} \geq 1 + \sum_{i=1}^{m-1} x_{ij}$, for all j .*

Theorem 14. $t(K_1 \# \dots \# K_n) \geq n$.

Proof. Here

$$\sum_{j=1}^n \left(\sum_{i=1}^m y_{ij} \right) \geq \sum_{j=1}^n \left(1 + \sum_{i=1}^{m-1} x_{ij} \right) = n + \sum_{j=1}^n \sum_{i=1}^{m-1} x_{ij}.$$

Hence,

$$\sum_{j=1}^n \sum_{i=1}^m y_{ij} - \sum_{j=1}^n \sum_{i=1}^{m-1} x_{ij} \geq n.$$

Thus,

$$\sum_{j=1}^n \sum_{i=1}^m -2(y_{ij}) - \sum_{j=1}^n \sum_{i=1}^{m-1} -2(x_{ij}) \leq -2n$$

and by definition,

$$\sum_{j=1}^n \sum_{i=1}^m \chi(S_i \cap C(K_j)) - \sum_{j=1}^n \sum_{i=1}^{m-1} \chi(F_i \cap C(K_j)) \leq -2n.$$

So,

$$\chi(S) = \sum_{i=1}^m \chi(S_i) - \sum_{i=1}^{m-1} \chi(F_i) \leq -2n.$$

Whence

$$\text{genus}(S) \geq n + 1$$

and

$$t(K_1 \# \dots \# K_n) \geq n.$$

\square

REFERENCES

- [1] *Knots*
G. Burde, H. Zieschang,
de Gruyter, Studies in Mathematics 5, Berlin, New York
- [2] *3-manifolds*
J. Hempel,
Annals of Math. Studies 86 (1976), Princeton University Press
- [3] *Lectures on Three-manifold Topology*
W. Jaco,
Regional Conference Series in Mathematics 43 (1981), Amer. Math. Soc.
- [4] *Problem's in Low-Dimensional Topology*
R. Kirby, in *Geometric Topology – Proceedings of the 1993 Georgia International Topology Conference*, ed. by W. H. Kazez, Vol. 2, Part 2, pp. 35–473
- [5] *Every two generator knot is prime*
F.H. Norwood, Proc. Amer. Math. Soc. 86 (1982), 143–147
- [6] *Knots and Links*
D. Rolfsen,
Publish or Perish, Inc. Houston, Texas
- [7] *Heegaard splittings of compact 3-manifolds*
M. Scharlemann,
to appear in the *Handbook of Geometric Topology*, ed. by R. Daverman and R. Sherr, Elsevier Press.
- [8] *Thin position for 3-manifolds*
M. Scharlemann, A. Thompson, AMS Contemporary Math. 164 (1994) 231–238
- [9] *Additivity of Tunnel Number*
J. Schultens,
preprint

DEPARTMENT OF MATHEMATICS, UCSB, SANTA BARBARA, CA 93106,

DEPARTMENT OF MATHEMATICS & CS, EMORY UNIVERSITY, ATLANTA, GA 30322

E-mail address: `mgscharl@math.ucsb.edu`, `jcs@mathcs.emory.edu`